

Uniqueness of differential polynomials sharing one value

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Abstract

We prove some uniqueness results which improve and generalize results of Jiang-Tao Li and Ping Li [*Uniqueness of entire functions concerning differential polynomials. Commun. Korean Math. Soc. 30 (2015), No. 2, pp. 93-101*].

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1 Introduction

Let f be a non-constant meromorphic function in the complex plane \mathbb{C} . We assume that the reader is familiar with the standard notions of the Nevanlinna value distribution theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$ (see e.g., [3]).

For $a \in \mathbb{C} \cup \{\infty\}$, we say that two meromorphic functions f and g share a CM, if $f - a$ and $g - a$ have the same set of zeros with same multiplicities, and if we do not consider the multiplicities then f and g are said to share a IM.

In [11], C.C. Yang posed the following question:

Question: What can be said about two entire functions f and g , when they share 0 CM and their derivatives share 1 CM ?

In 1990, Yi [4, 5], answered the above question by proving: *Let f and g be two non-constant entire functions such that f and g share 0 CM. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and $\delta(0, f) > 1/2$, where k is non-negative integer, then $f \equiv g$ unless $f^{(k)}.g^{(k)} \equiv 1$;* and for meromorphic functions he proved: *Let f and g be two non-constant meromorphic functions such that f and g share 0 and ∞ CM. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and $2\delta(0, f) + (k+2)\Theta(\infty, f) > k+3$, where k is non-negative integer, then $f \equiv g$ unless $f^{(k)}.g^{(k)} \equiv 1$.*

For a non-constant meromorphic function h , we denote by

$$L(h) = h^{(k)} + a_1 h^{(k-1)} + a_2 h^{(k-2)} + \dots + a_{k-1} h' + a_k h,$$

the differential polynomial of h , where a_1, a_2, \dots, a_k are finite complex numbers and k is a positive integer. We denote the order and lower order of h by $\lambda(h)$ and $\mu(h)$, respectively. Also by $\sigma(h)$ and $\sigma(1/h)$, we denote the exponent of convergence of zeros and poles of h respectively.

Recently, Jiang-Tao Li and Ping Li [2] generalized first result of Yi(as stated above) for entire fuctions as

Theorem A. Let f and g be two non-constant entire functions such that f and g share 0 CM. Suppose $L(f)$ and $L(g)$ share 1 CM and $\delta(0, f) > 1/2$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $L(f).L(g) \equiv 1$.

Theorem B. Let f and g be two non-constant entire functions such that f and g share 0 CM. Suppose $L(f)$ and $L(g)$ share 1 IM and $\delta(0, f) > 4/5$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $L(f).L(g) \equiv 1$.

We recall the following definition of weighted sharing:

Definition 1.1. Let f and g be two non constant meromorphic functions and k be a non-negative integer or ∞ . For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f and g share the value a with weight k .

We write “ f and g share (a, k) ” to mean that “ f and g share the value a with weight k ”. Clearly if f and g share (a, k) , then f and g share (a, p) , $0 \leq p < k$. Also we note that f and g share the value a IM(ignoring multiplicity) or CM(counting multiplicity) if and only if f and g share $(a, 0)$ or (a, ∞) , respectively.

Definition 1.2. *let f and g share 1 IM, and let z_0 be a zero of $f-1$ with multiplicity p and a zero of $g-1$ with multiplicity q . We denote by $N_E^{(1)}(r, 1/(f-1))$, the counting function of the zeros of $f-1$ when $p = q = 1$. By $\overline{N}_E^{(2)}(r, 1/(f-1))$, we denote the counting function of the zeros of $f-1$ when $p = q \geq 2$ and by $\overline{N}_L(r, 1/(f-1))$, we denote the counting function of the zeros of $f-1$ when $p > q \geq 1$, each point in these counting functions is counted only once; similarly, the terms $N_E^{(1)}(r, 1/(g-1))$, $\overline{N}_E^{(2)}(r, 1/(g-1))$ and $\overline{N}_L(r, 1/(g-1))$. Also, we denote by $\overline{N}_{f>k}(r, 1/(g-1))$, the reduced counting function of those zeros of $f-1$ and $g-1$ such that $p > q = k$, and similarly the term $\overline{N}_{g>k}(r, 1/(f-1))$.*

With the help of weighted sharing, we generalize Theorem A and Theorem B as

Theorem 1.3. *Let f and g be two non-constant entire functions such that f and g share 0 CM. Suppose $L(f)$ and $L(g)$ share $(1, l)$, $l \geq 0$ with one of the following conditions:*

- (i) $l \geq 2$ and $\delta(0, f) > 1/2$
- (ii) $l = 1$ and $\delta(0, f) > 3/5$
- (iii) $l = 0$ and $\delta(0, f) > 4/5$.

If $\lambda(f) \neq 1$, then $f \equiv g$ unless $L(f).L(g) \equiv 1$.

For meromorphic functions, we prove the following result:

Theorem 1.4. *Let f and g be two non-constant meromorphic functions of finite order such that f and g share 0 and ∞ CM. Suppose $L(f)$ and $L(g)$ share $(1, l)$, $l \geq 0$ with one of the following conditions:*

- (i) $l \geq 2$ and

$$(k+2)\Theta(\infty, f) + 2\delta(0, f) > k+3 \quad (1.1)$$

- (ii) $l = 1$ and

$$(3k+5)\Theta(\infty, f) + 5\delta(0, f) > 3k+9 \quad (1.2)$$

- (iii) $l = 0$ and

$$(4k+5)\Theta(\infty, f) + 5\delta(0, f) > 4k+9 \quad (1.3)$$

If $\lambda(f) \neq 1$ and $\sigma(1/f) \leq \sigma(f)$, then $f \equiv g$ unless $L(f).L(g) \equiv 1$.

The main tool of our investigations in this paper is Nevanlinna value distribution theory of meromorphic functions(see [3]).

2 Proof of the Main Result

We shall use the following results in the proof of our main result:

Lemma 2.1. [2] *Let f be a non-constant meromorphic function and k be a non-negative integer. Then*

$$T(r, L(f)) \leq T(r, f) + k\overline{N}(r, f) + S(r, f). \quad (2.1)$$

Lemma 2.2. [2] *Let f be a non-constant meromorphic function and a be a meromorphic function such that $T(r, a) = o(T(r, f))$ as $r \rightarrow \infty$. If f is not a polynomial, then*

$$N\left(r, \frac{1}{L(f) - L(a)}\right) \leq T(r, L(f)) - T(r, f) + N\left(r, \frac{1}{f - a}\right) + S(r, f) \quad (2.2)$$

and

$$N\left(r, \frac{1}{L(f) - L(a)}\right) \leq N\left(r, \frac{1}{f - a}\right) + k\overline{N}(r, f) + S(r, f). \quad (2.3)$$

Lemma 2.3. [1] *Let f and g be two non-constant meromorphic functions.*

(i) *If f and g share $(1, 0)$, then*

$$\overline{N}_L\left(r, \frac{1}{f - 1}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r), \quad (2.4)$$

where $S(r) = o(T(r))$ as $r \rightarrow \infty$ with $T(r) = \max\{T(r, f); T(r, g)\}$.

(ii) *If f and g share $(1, 1)$, then*

$$\begin{aligned} 2\overline{N}_L\left(r, \frac{1}{f - 1}\right) + 2\overline{N}_L\left(r, \frac{1}{g - 1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{f - 1}\right) - \overline{N}_{f>2}\left(r, \frac{1}{g - 1}\right) \\ \leq N\left(r, \frac{1}{g - 1}\right) - \overline{N}\left(r, \frac{1}{g - 1}\right). \end{aligned} \quad (2.5)$$

Lemma 2.4. [10] *Suppose f_j ($j = 1, 2, \dots, n + 1$) and g_j ($j = 1, 2, \dots, n$) ($n \geq 1$) are entire functions satisfying the following conditions:*

$$(i) \sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv f_{n+1}(z),$$

(ii) *The order of $f_j(z)$ is less than the order of $e^{g_k(z)}$ for $1 \leq j \leq n + 1$, $1 \leq k \leq n$. And furthermore, the order of $f_j(z)$ is less than the order of $e^{g_h(z) - g_k(z)}$ for $n \geq 2$ and $1 \leq j \leq n + 1$, $1 \leq h < k \leq n$.*

Then $f_j \equiv 0$ ($j = 1, 2, \dots, n + 1$).

Lemma 2.5. [10] *Suppose f_j ($j = 1, 2, \dots, n$) are meromorphic functions and g_j ($j = 1, 2, \dots, n$) ($n \geq 2$) are entire functions satisfying the following conditions:*

$$(i) \sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv 0.$$

$$(ii) g_j(z) - g_k(z) \text{ are non-constants for } 1 \leq j < k \leq n.$$

$$(iii) \text{ For } 1 \leq j \leq n, 1 \leq h < k \leq n,$$

$$T(r, f_j) = o(T(r, e^{g_h - g_k})),$$

as $r \rightarrow \infty$. Then $f_j(z) \equiv 0$ ($j = 1, 2, \dots, n$).

Lemma 2.6. [10] If $h(z)$ be a polynomial of degree p and $f(z) = e^{h(z)}$, then $\lambda(f) = \mu(f) = p$.

Lemma 2.7. [10] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions in the complex plane. If $\lambda(f) < \mu(g)$, then $T(r, f) = o(T(r, g))$ as $r \rightarrow \infty$.

We only prove Theorem 1.4 as the proof of Theorem 1.3 follows on the similar lines.

Proof of Theorem 1.4: First we assume that $L(f) \equiv c$, a finite constant. Then f has to be entire and

$$f \equiv c_1 + \sum_{i=1}^m p_i(z) e^{\alpha_i z},$$

where c_1 is finite constant, $m(\leq k)$ is a positive integer, α_i are distinct complex numbers and $p_i(z)$ are polynomials ($i = 1, 2, \dots, m$).

Since $\lambda(f) \neq 1$, we get $\lambda(f) < 1$ and so $e^{\alpha_i z}$ is constant. Thus f is a polynomial and so $\delta(0, f) = 0$, which contradicts (1.1), (1.2) and (1.3).

Assume that both $L(f)$ and $L(g)$ are non-constant. Since f and g share 0 and ∞ CM, and $L(f)$ and $L(g)$ share $(1, l)$, it follows from Milloux's inequality and (2.3)

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + N(r, \frac{1}{f}) + \overline{N}\left(r, \frac{1}{L(f) - 1}\right) + S(r, f) \\ &= \overline{N}(r, g) + N(r, \frac{1}{g}) + \overline{N}\left(r, \frac{1}{L(g) - 1}\right) + S(r, f) \\ &\leq 2T(r, g) + k\overline{N}(r, g) + N\left(r, \frac{1}{g - 1}\right) + S(r, f) + S(r, g) \\ &\leq (k + 3)T(r, g) + S(r, f) + S(r, g). \end{aligned}$$

Similarly

$$T(r, g) \leq (k + 3)T(r, f) + S(r, f) + S(r, g).$$

Thus $S(r, f) = S(r, g)$ and $\lambda(f) = \lambda(g)$.

Let $F = L(f)$ and $G = L(g)$. Then F and G share $(1, l)$, $l \geq 0$. Define

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right). \quad (2.6)$$

Assume that $H \not\equiv 0$. Then from (2.6), we have

$$m(r, H) = S(r, F) + S(r, G).$$

By the Second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r, F) + T(r, G) &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{G-1}\right) - N_0\left(r, \frac{1}{F'}\right) - N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G) \\ &= 2\overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{G-1}\right) - N_0\left(r, \frac{1}{F'}\right) - N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G), \end{aligned} \quad (2.7)$$

where $N_0(r, 1/F')$ denotes the counting function of the zeros of F' which are not the zeros of $F(F-1)$ and $N_0(r, 1/G')$ denotes the counting function of the zeros of G' which are not the zeros of $G(G-1)$.

We consider the following cases:

Case (i). If $l \geq 1$, then from (2.6), we have

$$\begin{aligned} N_E^{(1)}\left(r, \frac{1}{F-1}\right) &\leq N\left(r, \frac{1}{H}\right) + S(r, F) + S(r, G) \\ &\leq T(r, H) + S(r, F) + S(r, G) \\ &= N(r, H) + S(r, F) + S(r, G) \\ &\leq \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) \\ &\quad + \overline{N}_L\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G) \end{aligned}$$

and so

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &= N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) \\
&\quad + \overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G) \\
&\leq \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) \\
&\quad + 2\overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\
&\quad + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G).
\end{aligned} \tag{2.8}$$

Subcase 1.1: When $l = 1$. Then we have

$$\overline{N}_L\left(r, \frac{1}{F-1}\right) \leq \frac{1}{2}N\left(r, \frac{1}{F'} \mid F \neq 0\right) \leq \frac{1}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right), \tag{2.9}$$

where $N\left(r, \frac{1}{F'} \mid F \neq 0\right)$ denotes the zeros of F' , that are not the zeros of F .

From (2.5) and (2.9), we have

$$\begin{aligned}
2\overline{N}_L\left(r, \frac{1}{F-1}\right) + 2\overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\
\leq N\left(r, \frac{1}{G-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) + S(r, F) + S(r, G) \\
\leq N\left(r, \frac{1}{G-1}\right) + \frac{1}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + S(r, F) + S(r, G).
\end{aligned} \tag{2.10}$$

Thus, from (2.8) and (2.10), we have

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &\leq \frac{1}{2}\overline{N}(r, F) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) \\
&\quad + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G-1}\right) \\
&\quad + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G) \\
&\leq \frac{1}{2}\overline{N}(r, F) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) \\
&\quad + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + T(r, G) \\
&\quad + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G).
\end{aligned} \tag{2.11}$$

From (2.2), (2.3), (2.7) and (2.11), we obtain

$$\begin{aligned}
T(r, F) &\leq \frac{5}{2}\overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) \\
&\quad + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + S(r, F) + S(r, G) \\
&\leq \frac{5}{2}\overline{N}(r, F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + S(r, F) + S(r, G) \\
&= \frac{5}{2}\overline{N}(r, F) + N\left(r, \frac{1}{L(f)}\right) + \frac{1}{2}\overline{N}\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(g)}\right) + S(r, F) + S(r, G) \\
&\leq \frac{5}{2}\overline{N}(r, f) + T(r, L(f)) - T(r, f) + N\left(r, \frac{1}{f}\right) + \frac{1}{2}N\left(r, \frac{1}{f}\right) \\
&\quad + \frac{k}{2}\overline{N}(r, f) + N\left(r, \frac{1}{g}\right) + k\overline{N}(r, g) + S(r, f) + S(r, g) \\
&= T(r, L(f)) - T(r, f) + \left(\frac{3k+5}{2}\right)\overline{N}(r, f) + \frac{5}{2}N\left(r, \frac{1}{f}\right) + S(r, f).
\end{aligned}$$

That is,

$$2T(r, f) \leq (3k+5)\overline{N}(r, f) + 5N\left(r, \frac{1}{f}\right) + S(r, f),$$

and so $(3k+5)\Theta(\infty, f) + 5\delta(0, f) \leq 3k+8$, a contradiction to (1.2).

Subcase 1.2: When $l \geq 2$.

In this case, we have

$$\begin{aligned}
2\overline{N}_L\left(r, \frac{1}{F-1}\right) + 2\overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\
\leq N\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G).
\end{aligned}$$

Thus from (2.8), we get

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &\leq \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{G-1}\right) \\
&\quad + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G) \\
&\leq \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + T(r, G) \\
&\quad + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G).
\end{aligned} \tag{2.12}$$

Since f and g share 0 and ∞ CM, from (2.2), (2.3), (2.7) and (2.12), we obtain

$$\begin{aligned}
T(r, F) &\leq 2\overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G) \\
&\leq 2\overline{N}(r, f) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G) \\
&= 2\overline{N}(r, f) + N\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(g)}\right) + S(r, f) + S(r, g) \\
&\leq 2\overline{N}(r, f) + T(r, L(f)) - T(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + k\overline{N}(r, g) + S(r, f) + S(r, g) \\
&= T(r, L(f)) - T(r, f) + (k+2)\overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + S(r, f).
\end{aligned}$$

That is,

$$T(r, f) \leq (k+2)\overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + S(r, f),$$

and so $(k+2)\Theta(\infty, f) + 2\delta(0, f) \leq k+3$, a contradiction to (1.1).

Case (ii). If $l = 0$, then we have

$$N_E^{(1)}\left(r, \frac{1}{F-1}\right) = N_E^{(1)}\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G),$$

$$\overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) = \overline{N}_E^{(2)}\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G),$$

and also from (2.6), we have

$$\begin{aligned}
\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) &\leq N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) \\
&\quad + \overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G) \\
&\leq N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) + N\left(r, \frac{1}{G-1}\right) \\
&\quad + S(r, F) + S(r, G) \\
&\leq \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) \\
&\quad + \overline{N}_L\left(r, \frac{1}{G-1}\right) + N\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) \\
&\quad + N_0\left(r, \frac{1}{G'}\right) + S(r, F) + S(r, G). \tag{2.13}
\end{aligned}$$

From (2.2), (2.3), (2.4), (2.7) and (2.13), we obtain

$$\begin{aligned}
T(r, F) &\leq 2\overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}_{(2)}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G}\right) \\
&\quad + 2\overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G) \\
&\leq 2\overline{N}(r, F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}(r, F) \\
&\quad + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, G) + S(r, F) + S(r, G) \\
&\leq 5\overline{N}(r, f) + N\left(r, \frac{1}{L(f)}\right) + 2N\left(r, \frac{1}{L(f)}\right) + 2N\left(r, \frac{1}{L(g)}\right) + S(r, F) + S(r, G) \\
&\leq 5\overline{N}(r, f) + T(r, L(f)) - T(r, f) + N\left(r, \frac{1}{f}\right) + 2N\left(r, \frac{1}{f}\right) + 2k\overline{N}(r, f) \\
&\quad + 2N\left(r, \frac{1}{g}\right) + 2k\overline{N}(r, g) + S(r, f) + S(r, g) \\
&\leq T(r, L(f)) - T(r, f) + (4k+5)\overline{N}(r, f) + 5N\left(r, \frac{1}{f}\right) + S(r, f).
\end{aligned}$$

That is,

$$T(r, f) \leq (4k+5)\overline{N}(r, f) + 5N\left(r, \frac{1}{f}\right) + S(r, f),$$

and so $(4k+5)\Theta(\infty, f) + 5\delta(0, f) \leq 4k+9$, a contradiction to (1.3).

Thus our supposition is wrong and hence $H \equiv 0$. So (2.6) implies that

$$\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1},$$

and so we obtain

$$\frac{1}{F-1} = \frac{C}{G-1} + D, \quad (2.14)$$

where $C \neq 0$ and D are constants.

Here, the following three cases can arise:

Case(a) : When $D \neq 0, -1$. We rewrite (2.14) as

$$\frac{G-1}{C} = \frac{F-1}{D+1-DF},$$

we have

$$\overline{N}(r, G) = \overline{N}\left(r, \frac{1}{F - (D+1)/D}\right).$$

By Second fundamental theorem of Nevanlinna and (2.2), we have

$$\begin{aligned}
T(r, L(f)) &= T(r, F) + S(r, f) \\
&\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - (D+1)/D}\right) + S(r, f) \\
&\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, G) + S(r, f) \\
&\leq N\left(r, \frac{1}{L(f)}\right) + 2\overline{N}(r, f) + S(r, f) \\
&\leq T(r, L(f)) - T(r, f) + 2\overline{N}(r, f) + N(r, \frac{1}{f}) + S(r, f).
\end{aligned}$$

Thus

$$T(r, f) \leq 2\overline{N}(r, f) + N(r, \frac{1}{f}) + S(r, f),$$

and so $2\Theta(\infty, f) + \delta(0, f) \leq 2$, which contradicts (1.1), (1.2) and (1.3).

Case(b) : When $D = 0$. Then from (2.14), we have

$$G = CF - (C - 1). \quad (2.15)$$

So if $C \neq 1$, then

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - (C-1)/C}\right).$$

Since f and g share 0 and ∞ CM, by Second fundamental theorem of Nevanlinna, (2.2) and (2.3) gives

$$\begin{aligned}
T(r, L(f)) &= T(r, F) + S(r, f) \\
&\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - (C-1)/C}\right) + S(r, f) \\
&= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, \frac{1}{G}) + S(r, f) \\
&\leq \overline{N}(r, f) + N\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(g)}\right) + S(r, f) \\
&\leq \overline{N}(r, f) + T(r, L(f)) - T(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{g}) + k\overline{N}(r, g) + S(r, f) \\
&= T(r, L(f)) - T(r, f) + (k+1)\overline{N}(r, f) + 2N(r, \frac{1}{f}) + S(r, f).
\end{aligned}$$

Thus

$$T(r, f) \leq (k+1)\overline{N}(r, f) + 2N(r, \frac{1}{f}) + S(r, f),$$

and so $(k+1)\Theta(\infty, f) + 2\delta(0, f) \leq k+2$, which contradicts (1.1), (1.2) and (1.3).

Thus, $C = 1$ and so in this case from (2.15), we obtain $F \equiv G$ and so

$$L(f) \equiv L(g).$$

Case(c) : When $D = -1$. Then from (2.14) we have

$$\frac{1}{F-1} = \frac{C}{G-1} - 1. \quad (2.16)$$

So if $C \neq -1$, then

$$\overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - C/(C+1)}\right).$$

Since f and g share 0 and ∞ CM, by Second fundamental theorem of Nevanlinna, (2.2) and (2.3), we have

$$\begin{aligned} T(r, L(f)) &= T(r, F) + S(r, f) \\ &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - C/(C+1)}\right) + S(r, f) \\ &= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + N\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(g)}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + T(r, L(f)) - T(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + k\overline{N}(r, g) + S(r, f) \\ &= T(r, L(f)) - T(r, f) + (k+1)\overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned}$$

Thus

$$T(r, f) \leq (k+1)\overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + S(r, f),$$

and so $(k+1)\Theta(\infty, f) + 2\delta(0, f) \leq k+2$, which contradicts (1.1), (1.2) and (1.3).

Thus, $C = -1$ and so in this case from (2.16), we obtain $FG \equiv 1$ and so $L(f)L[g] = 1$.

If $L(f) \equiv L(g)$, then $L(f-g) \equiv 0$ and so $f-g$ has to be entire and we have (see [8])

$$f - g = \sum_{i=1}^m p_i(z) e^{\alpha_i z},$$

where $m(\leq k)$ is a positive integer, α_i are distinct complex numbers and $p_i(z)$ are polynomials ($i = 1, 2, \dots, m$).

Thus

$$\lambda(f-g) = \lambda\left(\sum_{i=1}^m p_i(z) e^{\alpha_i z}\right) \leq 1.$$

We consider the following cases:

Case (i). When $\lambda(f) < 1$. Since f and g share 0 and ∞ CM, we have $f/g = e^{h(z)}$, where $h(z)$ is an entire function. Also as $\lambda(f) = \lambda(g)$, we have

$$\lambda(e^{h(z)}) = \lambda(f/g) \leq \max\{\lambda(f), \lambda(1/g)\} < 1.$$

Thus $e^{h(z)}$ is a constant, say c and so $f \equiv cg$ which implies that $L(f) \equiv cL(g)$. But $L(f) \equiv L(g)$, so we get $c = 1$ and thus $f \equiv g$.

Case (ii). When $\lambda(f) > 1$. Since f and g are meromorphic functions of finite order, by Hadamard's factorization theorem we have

$$f(z) = \frac{P(z)}{Q(z)}e^{l_1(z)} \quad \text{and} \quad g(z) = \frac{P(z)}{Q(z)}e^{l_2(z)},$$

where $P(z)$ is the canonical product formed with the common zeros of f and g , $Q(z)$ is the canonical product formed with the common poles of f and g , and l_1, l_2 are the polynomials of degree less than or equal to $\lambda(f), \lambda(g)$ respectively. Thus

$$f - g = \frac{P(z)}{Q(z)}e^{l_1(z)} - \frac{P(z)}{Q(z)}e^{l_2(z)},$$

or we can write

$$\frac{P(z)}{Q(z)}e^{l_1(z)} - \frac{P(z)}{Q(z)}e^{l_2(z)} - (f - g)e^{l_3(z)} \equiv 0, \quad (2.17)$$

where $l_3(z) \equiv 0$.

Also

$$\lambda(P) = \sigma(f) \leq \sigma(f - g) \leq \lambda(f - g) \leq 1,$$

and since $\sigma(1/f) \leq \sigma(f)$, we have

$$\lambda(Q) = \sigma(1/f) \leq \sigma(f) \leq \sigma(f - g) \leq \lambda(f - g) \leq 1.$$

Thus

$$\lambda\left(\frac{P}{Q}\right) \leq \max\{\lambda(P), \lambda(Q)\} \leq 1.$$

Since $f - g = (e^{l_1 - l_2})g$ and $\lambda(f) = \lambda(g) > 1$, we have $\lambda(e^{l_1}) > 1$, $\lambda(e^{l_1}) > 1$ and $\lambda(e^{l_1 - l_2}) > 1$, and so $\lambda(e^{l_i - l_j}) > 1$, where $1 \leq i < j \leq 3$. Thus $l_i - l_j$ is non-constant, where $1 \leq i < j \leq 3$ and by lemma 2.6 and 2.7, we get

$$T(r, f - g) = o(T(r, e^{l_i - l_j})) \quad \text{and} \quad T(r, P/Q) = o(T(r, e^{l_i - l_j})),$$

as $r \rightarrow \infty$. Thus by lemma(2.5), we have $P/Q \equiv 0$ and $f - g \equiv 0$ which implies that $f(z) \equiv 0$, which is a contradiction. So $l_1 = l_2$ and hence $f \equiv g$. \square

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